M.Math. IInd year
First semestral exam 2020
Number Theory — Instructor : B.Sury
December 26, 2020 — 10 AM - 1 PM
Answer ANY SIX - parts of questions will not be accepted.

Q 1.
If $d > 2$, show that the ring $\mathbb{Z}[\sqrt{-d}]$ cannot be a Euclidean domain.

OR

Let q be a prime of the form $4k+1$ such that $p = 2q+1$ is also prime. Prove that 2 is a primitive root modulo p but may not be a primitive root modulo q.

Q 2.
Let p be an odd prime. Show that the numerator of $\frac{n^p-n}{p}+\sum_{r=1}^{p-1}\frac{(-1)^r(1^r+2^r+\cdots+(n-1)^r)}{r}$ is a multiple of p.

OR

Let $p > 3$ be a prime and write $\sum_{d=1}^{p}\frac{1}{d} = \frac{r}{ps}$. Prove that p^3 divides $r - s$.

Q 3.
Let a, b, x_0 be positive integers and define $x_n = ax_{n-1} + b$ for $n > 0$. Prove that not every x_n can be prime.

OR

Prove that a Fermat number $2^{2^n} + 1$ can never be a prime power p^r with $r \geq 2$.

Q 4.
If p is a prime and $n > r$ are positive integers, write $n = \sum_{i \geq 0} n_i p^i$ and $r = \sum_{i \geq 0} r_i p^i$ be their base p expansions. Prove that

$$\binom{n}{r} \equiv \prod_i \binom{n_i}{r_i} \quad (mod \ \ p).$$

OR

Prove that $(p-2)! = 1 + p^n$ does not have any solution in primes $p > 5$ and positive integers n.

Q 5.
Evaluate the sum of Legendre symbols $\sum_{a=1}^{p-1} \left(\frac{a(a+1)}{p} \right)$.

OR

Let $p > 3$ be a prime. Show that $\left(\frac{3}{p} \right) = \prod_{r=1}^{(p-1)/2} (3 - 4\sin^2(2\pi r/p))$.

Q 6. Prove:
(i) $\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\phi(d)}$.
(ii) $\mu(n) = \sum_{(k,n)=1} e^{2ik\pi/n}$.

OR

Prove:
(i) $\prod_{(a,n)=1} a = n^{\phi(n)} \prod_{d|n} (d!/d^d)^{\mu(n/d)}$.
(ii) If $m \geq 1$ and n has more than m distinct prime factors, then $\sum_{d|n} \mu(d) \log(d)^m = 0$.

Q 7.
Observe that no isosceles right angled triangle can have all sides of integer length. However, show that for any $\epsilon > 0$, there are primitive Pythagorean triples for which the corresponding right angled triangles have the acute angles within ϵ of 45 degrees.

OR

Prove that $(x^2-17)(x^2-19)(x^2-323) \equiv 0 \pmod{n}$ has a solution for every positive integer n.

Q 8.
(i) Show that if n is represented by a quadratic form f of discriminant d, then $4an$ is a square mod $|d|$.
(ii) If $4|d$ and d is not a perfect square, and every form of discriminant d is primitive, show that $d/4$ is square-free and congruent to 2 or 3 modulo 4.

OR

(i) Give an example (with proof) of forms $ax^2+bxy+cy^2$ and $ax^2-bxy+cy^2$ that are not equivalent.
(ii) Determine all reduced forms of discriminant -19. Reduce the quadratic form $7x^2+25xy+23y^2$ to its reduced form.

Q 9.
Obtain simple, continued fraction expansions of $\sqrt{13}$ and $\sqrt{23}$. Deduce a solution of $x^2-13y^2=-1$. Does the analogous equation $x^2-23y^2=-1$ have a solution?

OR

Using Bertrand's postulate or otherwise, prove:
(i) For any positive integer n, prove that $\{1, 2, \cdots, 2n\}$ is a union of n pairs each of whose sums is a prime.
(ii) $n! = m^k$ does not have solutions for $m, n, k > 1$.